

DELOOPS OF THE SPACES OF LONG EMBEDDINGS

KEIICHI SAKAI

ABSTRACT. The homotopy fiber of the inclusion from the long embedding space into the long immersion space is known to be an iterated based loop space (if the codimension is greater than two). In this short paper we deloop the homotopy fiber to obtain the topological Stiefel manifold, combining results of Lashof and of Lees. We also give a deloop of the long embedding space, which can be regarded as a version of Morlet-Burghlelea-Lashof's deloop of the diffeomorphism group of the disk relative to the boundary. As a corollary, we show that in the stable range of dimensions the homotopy fiber is weakly equivalent to a space on which the framed little disks operad acts, and hence its rational homology is a (higher) BV algebra.

1. INTRODUCTION

Let $E^d = E_{n,j}^d$ (resp. $I^d = I_{n,j}^d$) be the space of *long embeddings* (resp. *long immersions*), that is, smooth embeddings $f : \mathbb{R}^j \hookrightarrow \mathbb{R}^n$ (resp. immersions $\mathbb{R}^j \looparrowright \mathbb{R}^n$) such that $f(x) = (x, \mathbf{0})$ if $|x| \geq 1$ (“ d ” indicates that we are considering differentiable maps). We also consider the space $fE_{n,j}^d$ ($fI_{n,j}^d$) of *framed* long embeddings (immersions) $\mathbb{R}^j \times (-\epsilon, \epsilon)^{n-j} \rightarrow \mathbb{R}^n$. Budney [1] defined an action of little $(j+1)$ -disks operad \mathcal{C}_{j+1} on (a space equivalent to) $fE_{n,j}^d$. Consequently $fE_{n,j}^d$ ($n-j \geq 3$) turns out to be weakly equivalent to a $(j+1)$ -fold loop space by the loop space recognition principle [14]. Budney's \mathcal{C}_{j+1} -action also applies to $fI_{n,j}^d$ in such a way that the inclusion $fE_{n,j}^d \rightarrow fI_{n,j}^d$ is a map of \mathcal{C}_{j+1} -spaces. Thus the space $\overline{E}_{n,j}^d$, the homotopy fiber of $fE_{n,j}^d \rightarrow fI_{n,j}^d$ (or equivalently of $E_{n,j}^d \rightarrow I_{n,j}^d$), is also a \mathcal{C}_{j+1} -space and hence a $(j+1)$ -fold loop space if $n-j \geq 3$ (this argument is same as the proof of [22, Proposition 1.1]). Sinha [21] also proved that $\overline{E}_{n,1}^d$ ($n \geq 4$) is weakly equivalent to a double loop space by a cosimplicial method, and based on Sinha's work, Salvatore [19] showed that $E_{n,1}$ ($n \geq 4$) is a double loop space so that there are double loop maps $E_{n,1} \rightarrow fE_{n,1}$ and $E_{n,1} \rightarrow \overline{E}_{n,1}$.

A natural question is; what the deloop of $\overline{E}_{n,j}^d$ (and of $fE_{n,j}^d$) is. Dwyer-Hess [7] and Tourtchine [23] described a deloop of $\overline{E}_{n,1}^d$ as the derived space of maps between some operads. The purpose of this short paper is to give a simple deloop of $\overline{E}_{n,j}^d$ which has already appeared implicitly in Lashof's paper [12].

Theorem 1.1. *If $n-j \geq 3$ and $n \geq 5$, then $\overline{E}_{n,j}^d$ is weakly equivalent to the $(j+1)$ -fold based loop space of the topological Stiefel manifold $V_{n,j}^t$.*

The *topological Stiefel manifold* $V_{n,j}^t$ (it is not a manifold in the usual sense) is defined to be the orbit space $\text{Top}(n)/\text{Top}(n,j)$, where $\text{Top}(n,j)$ is the topological

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group of germs at $\mathbf{0}$ of homeomorphisms $\mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$ which restrict to the identity on $\mathbb{R}^j \times \{0\}^{n-j}$, and $\text{Top}(n) := \text{Top}(n, 0)$.

Since the orthogonal group acts on the topological Stiefel manifold, the framed recognition principle [20, Theorem 3.1] implies the following.

Corollary 1.2. *If $n - j \geq 3$, $n \geq 5$ and $n \geq 2j + 1$, then $\overline{E}_{n,j}^d$ is weakly equivalent to a space on which the framed $(j + 1)$ -disks operad $\tilde{\mathcal{C}}_{j+1}$ acts. Consequently $H_*(\overline{E}_{n,j}^d; \mathbb{Q})$ is a BV_{j+1} -algebra [20, Definition 5.2].*

It is well known (though not so frequently mentioned) that $E_{n,j}^d$ ($n - j \geq 3$) can be delooped j times, because $\pi_0 E_{n,j}^d$ is a group if $n - j \geq 3$ [9] and the little j -disks operad acts on $E_{n,j}^d$ in a similar fashion to the case of j -fold based loop spaces. We can also describe a deloop of $E_{n,j}^d$.

Proposition 1.3. *If $n - j \geq 3$ and $n \geq 5$, then $E_{n,j}^d$ is weakly equivalent to $\Omega^j V_{n,j}^{t/d}$, where $V_{n,j}^{t/d}$ is the homotopy fiber of the map from the (usual) Stiefel manifold $V_{n,j}^d$ to $V_{n,j}^t$.*

Notice that the deloop in Proposition 1.3 can be regarded as a “positive codimension version” of Morlet-Burghelea-Lashof’s deloop of the diffeomorphism group $\text{Diff}(D^n, \partial) = E_{n,n}^d$ of the disk relative to the boundary [4, 17];

$$(1.1) \quad \text{Diff}(D^n, \partial) \sim \Omega^{n+1}(\text{Top}(n)/O(n)).$$

Indeed (1.1) can be written as $E_{n,n}^d \sim \Omega^n V_{n,n}^{t/d}$, since $\text{Top}(n) = V_{n,n}^t$, $O(n) = V_{n,n}^d$, and $O(n) \rightarrow \text{Top}(n) \rightarrow \text{Top}(n)/O(n)$ is a fiber bundle [8, Theorem 4.1] and hence a Serre fibration.

The proof of the following is similar to that of Corollary 1.2.

Corollary 1.4. *If $n - j \geq 3$ and $n \geq 2j$, then $E_{n,j}^d$ is weakly equivalent to a space on which $\tilde{\mathcal{C}}_j$ acts. Consequently $H_*(E_{n,j}^d; \mathbb{Q})$ is a BV_j -algebra.*

Proposition 1.3 gives rise to an alternative proof of the useful fact which was proved in [2, Proposition 3.9 (1)] by means of a spinning method (in a wider range of dimensions).

Corollary 1.5 ([2]). *If $n - j \geq 3$ and $n \geq 5$, then $\pi_k E_{n,j}^d \cong \pi_0 E_{n+k,j+k}^d$ for $k \leq 2(n - j) - 5$.*

In §2 we prove of these results. In §3 some related questions are listed.

2. PROOFS

Let $E^t = E_{n,j}^t$ and $I^t = I_{n,j}^t$ be the spaces of locally flat topological long embeddings and immersions $\mathbb{R}^j \rightarrow \mathbb{R}^n$ respectively. Let $E^{t/d} = E_{n,j}^{t/d}$ and $I^{t/d} = I_{n,j}^{t/d}$ be the homotopy fibers of the inclusions $E^d \rightarrow E^t$ and $I^d \rightarrow I^t$ respectively.

Theorem 2.1 ([12, Theorem A (t/d)]). *If $n - j \geq 3$ and $n \geq 5$, then the map $E_{n,j}^{t/d} \rightarrow I_{n,j}^{t/d}$ is a weak homotopy equivalence.*

Remark 2.2. Theorem 2.1 was stated in [12] in terms of simplicial sets. As mentioned in [12, Appendix], by a work of Černavskiĭ [5], the simplicial sets of locally flat topological embeddings or immersions used in [12] are homotopy equivalent to

the singular complexes of our space $E_{n,j}^t$ or $I_{n,j}^t$ if the conditions on n and j are satisfied. Therefore we always assume $n - j \geq 3$ and $n \geq 5$ throughout this paper.

Proof of Theorem 1.1. Consider the following commutative diagram consisting of six fibration sequences;

$$(2.1) \quad \begin{array}{ccccc} \overline{E}^{t/d} & \longrightarrow & \overline{E}^d & \longrightarrow & \overline{E}^t \\ \downarrow & & \downarrow & & \downarrow \\ E^{t/d} & \longrightarrow & E^d & \longrightarrow & E^t \xrightarrow{\simeq} * \\ \downarrow (a) & & \downarrow & & \downarrow \\ I^{t/d} & \longrightarrow & I^d & \longrightarrow & I^t \end{array}$$

where \overline{E}^* is the homotopy fiber of $E^* \rightarrow I^*$, $*$ = $d, t, t/d$. Since by Theorem 2.1 the map (a) in (2.1) is a weak equivalence, $\overline{E}^{t/d}$ is weakly contractible and hence $\overline{E}^d \rightarrow \overline{E}^t$ is a weak equivalence.

On the other hand, since E^t is contractible by the Alexander trick ([12, p. 146, Example]), $\Omega I^t \rightarrow \overline{E}^t$ is a homotopy equivalence. Theorem 1.1 follows from Lees' topological Smale-Hirsch theorem $I_{n,j}^t \xrightarrow{\simeq} \Omega^j V_{n,j}^t$ [13]. \square

Remark 2.3. In fact Lees' theorem [13] asserts that there is a weak equivalence from the space $fI_{n,j}^t$ of topological framed long immersions $\mathbb{R}^j \times (-\epsilon, \epsilon)^{n-j} \looparrowright \mathbb{R}^n$ to $\Omega^j \text{Top}(n)$, which equivalence fits into the following diagram of fibration sequences;

$$\begin{array}{ccccc} \Omega^j \text{Top}(n, j) & \longrightarrow & fI_{n,j}^t & \longrightarrow & I_{n,j}^t \\ \parallel & & \sim \downarrow \text{Lees} & & \downarrow \\ \Omega^j \text{Top}(n, j) & \longrightarrow & \Omega^j \text{Top}(n) & \longrightarrow & \Omega^j V_{n,j}^t \end{array}$$

Thus we have $I_{n,j}^t \xrightarrow{\simeq} \Omega^j V_{n,j}^t$.

Remark 2.4. The above proof works even if the spaces of topological maps are replaced by those of piecewise-linear (PL) maps. In this case the proof relies on Haefliger-Poenaru's theorem $I_{n,j}^{PL} \xrightarrow{\simeq} \Omega^j V_{n,j}^{PL}$ [10] (the space of locally flat PL immersions and the PL Stiefel manifold V^{PL} are defined analogously to the topological case). In fact, if $n - j \geq 3$, then $V_{n,j}^{PL} \rightarrow V_{n,j}^t$ is a homotopy equivalence by [12, Proposition (t/pl)], and hence we may replace V^t with V^{PL} .

Proof of Proposition 1.3. As noted in [12, p. 146, Example], there is a fibration sequence

$$E^d \rightarrow I^d \rightarrow I^t.$$

This is because we can deduce a weak equivalence $E^d \xrightarrow{\simeq} I^{t/d}$ which makes (2.1) homotopy commutative, using Theorem 2.1 and the fact that E^t is contractible.

The weak equivalences $I_{n,j}^* \rightarrow \Omega^j V_{n,j}^*$ ($*$ = d, t) are both given as taking the germs of the long immersions. Thus $I_{n,j}^d \rightarrow I_{n,j}^t$ is equivalent to the j -fold loop map of $V_{n,j}^d \rightarrow V_{n,j}^t$ and hence the fiber E^d is equivalent to $\Omega^j V_{n,j}^{t/d}$. \square

Proof of Corollary 1.2. The framed recognition principle [20, Theorem 3.1] states that, in the homotopy category, the framed little $(j+1)$ -disks operad $\tilde{\mathcal{C}}_{j+1}$ acts on

a space X if and only if X is the $(j+1)$ -fold loop space of a j -connected based $SO(j+1)$ -space. Since $SO(n-j)$ acts on $V_{n,j}^t$ fixing the basepoint (the orbit of $\text{id}_{\mathbb{R}^n}$) and $V_{n,j}^t$ ($n-j \geq 3$) is $(n-j-1)$ -connected by [11, Theorem 3] and [12, Corollary 2 (t/pl)], the space $\overline{E}_{n,j}^d \sim \Omega^{j+1}V_{n,j}^t$ is acted on by \mathcal{C}_{j+1} if $n-j \geq j+1$ (hence $n \geq 2j+1$). The BV_{j+1} -structure on $H_*(\overline{E}_{n,j}^d; \mathbb{Q})$ is a consequence of the \mathcal{C}_{j+1} -action [20, Theorem 5.4, Example 5.5]. \square

Proof of Corollary 1.4. The proof is similar to that of Corollary 1.2; since $SO(n-j)$ acts on $V_{n,j}^*$ ($* = d, t$) preserving the basepoints ($[\text{id}_{\mathbb{R}^n}] \in V_{n,j}^*$), and $V_{n,j}^d \rightarrow V_{n,j}^t$ is $SO(n-j)$ -equivariant, the homotopy fiber $V_{n,j}^{t/d}$ is also acted on by $SO(n-j)$ preserving the basepoint (the constant path at $[\text{id}_{\mathbb{R}^n}] \in V_{n,j}^t$). Moreover $V_{n,j}^{t/d}$ is $2(n-j-2)$ -connected by (2.2) below and [15, Theorem A]. Thus $E_{n,j}^d \sim \Omega^j V_{n,j}^{t/d}$ is acted on by \mathcal{C}_j by the framed recognition principle [20] if $n-j \geq j$. \square

Proof of Corollary 1.5. Using Haefliger-Millett's theorem [16], Lashof proved in [12, Proposition (t/d)] that, when $n-j \geq 3$ and $m \leq 2n-j-5$, there is an isomorphism

$$(2.2) \quad \pi_m(V_{n,j}^{t/d}) \cong \pi_{m+1}(G, O, G_{n-j})$$

where G_q is the space of degree one maps $S^{q-1} \rightarrow S^{q-1}$ and G is the stable suspension (see [9]). On the other hand, by Haefliger's classification theorem [9],

$$(2.3) \quad \pi_{m+1}(G, O, G_q) \cong \pi_0 E_{m+q,m}^d$$

for $q \geq 3$. Proposition 1.3 and the isomorphisms (2.2) and (2.3) deduce

$$\pi_{m-j} E_{n,j}^d \stackrel{\text{Prop 1.3}}{\cong} \pi_m V_{n,j}^{t/d} \stackrel{(2.2)}{\cong} \pi_{m+1}(G, O, G_{n-j}) \stackrel{(2.3)}{\cong} \pi_0 E_{n-j+m,m}^d$$

for $n-j \geq 3$ and $j \leq m \leq 2n-j-5$. Putting $k = m-j$ completes the proof. \square

3. QUESTIONS

In [1, 3] the space $E_{3,1}^d$ is proved to be a free \mathcal{C}_2 -object, and hence $H_*(E_{3,1}^d; \mathbb{Q})$ is a free Poisson algebra [6]. An analogous result for \overline{E}^d for general n, j ($n-j \geq 3$) would be derived if the answer of the following question is affirmative.

Question 1. *Is $V_{n,j}^t$ a $(j+1)$ -fold suspension?*

Salvatore proved in [19] that $E_{n,1}^d$ ($n > 3$) is a double loop space. The following question asks whether the similar result holds for general n, j .

Question 2. *Is $V_{n,j}^{t/d}$ a based loop space so that there is any $(j+1)$ -fold loop map $\Omega^j V_{n,j}^{t/d} \rightarrow \Omega^{j+1} V_{n,j}^t$?*

Question 3. *How do the BV -structures of Corollary 1.2 and of [18] relate to each other?*

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DEPARTMENT OF MATHEMATICAL SCIENCES, SHINSHU UNIVERSITY, 3-1-1 ASAHI, MATSUMOTO,
NAGANO 390-8621, JAPAN

E-mail address: ksakai@math.shinshu-u.ac.jp

URL: <http://math.shinshu-u.ac.jp/~ksakai/index.html>